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MOTION AND FORCE EQUILIBRIUM OF A LOCAL PROCESS (A SOLITON) IN A CONTINUOUS FLUID MEDIUM

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Consideration is given to the motion and force equilibrium of a local process (a soliton) in a continuous fluid medium. Integral characteristics of a soliton are introduced. An equation of motion, a global equation of force equilibrium, and equations of force equilibrium along individual axes are obtained that include the integral characteristics of a soliton. These equations are shown to permit direct evaluation of the interrelationship of the most important parameters of a local process based on generalized information on its structure.

Introduction. Many actual processes in a continuous fluid medium (a liquid or a gas) are local, i.e., occur in a confined volume of space. Various factors provide long-term steady existence and force equilibrium of local processes (solitons). A common approach to the theoretical study of local processes is the investigation of evolution equations that correspond to the actual phenomena (waves on deep and shallow water, Rossby waves, etc. [1, 2]). However, many natural phenomena (as applied to the atmosphere: a hurricane, a tornado, and a thermal) are so intricate that their mathematical description is necessarily simplified and the equations are solved by numerical methods [3]. Therefore, it becomes necessary to evaluate local processes by the characteristic general parameters that are important in practical applications.

We considered the motion and force equilibrium of a steady local process as a single whole in a compressible continuous homogeneous fluid medium. Integral characteristics of the process over the volume of the region where it occurs were introduced. Interrelationships between these characteristics were obtained.

In investigation of local processes, solitons are sometimes singled out as bearers of special properties (for example, impact elasticity [1, 2]). Within the framework of the current work, these properties are insignificant and the terms "local process" and "soliton" will be used as synonyms.

Strongly and Weakly Localized Processes. The fluid far from the region that is involved in a local process is assumed to be motionless. Initially, we introduce an absolute coordinate system tied to the medium as a whole. Consideration is given to the pressure p, the mass density ρ, the velocity $\mathbf{v}$, and the volume density of the extraneous force $\mathbf{f}_s$ at each point of the absolute space (x, y, z). The extraneous force is assumed to be the sum of all forces (friction, gravity, etc.) except for the inertial force with the volume density $\mathbf{f}_i = \rho \frac{d\mathbf{v}}{dt}$ and the pressure force with the volume density $\mathbf{f}_p = -\text{grad}\, p$ [4, 5].

We assume that, at an infinitely large distance from the process localization region (PLR), the pressure p and the density ρ are invariable and equal to their undisturbed values $p = p_0$ and $\rho = \rho_0$, the fluid is motionless $\mathbf{v}_0 = 0$, and the extraneous force is absent $\mathbf{f}_{s0} = 0$.

For the differences of the fluid pressures and densities at an arbitrary point and in the free space we introduce the corresponding notation $p_\Delta \equiv p - p_0$, $\rho_\Delta \equiv \rho - \rho_0$. These differences are assumed to be small in comparison with their initial values:

$$|p_\Delta| \ll p_0\,, \quad |\rho_\Delta| \ll \rho_0\,. \tag{1}$$

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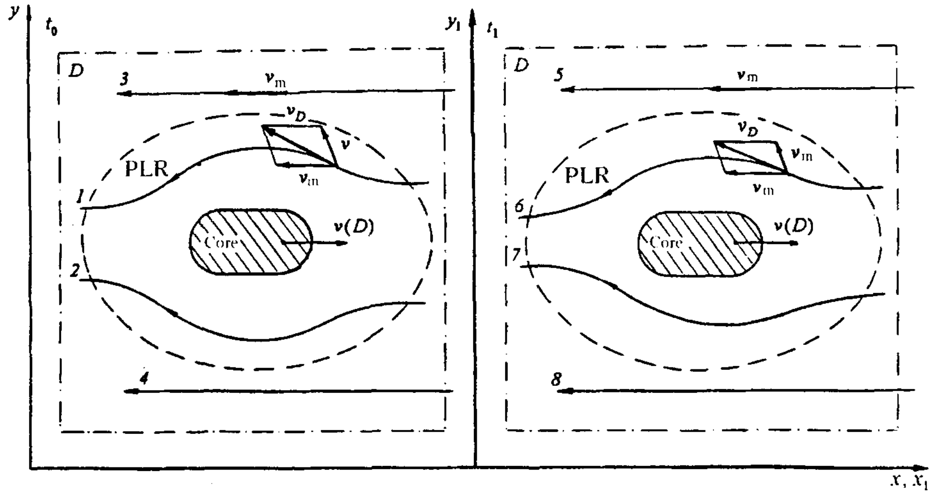

Fig. 1. Shift of a local steady process in a continuous fluid medium: x and y denote the position of the absolute and mobile axes at the instant of time t_0; x_1 and y_1 denote the absolute position of the mobile axes at the instant of time t_1; 1-4) streamlines in the mobile system at the instant of time t_0; 5-8) streamlines in the mobile system at the instant of time t_1.

As is well known [4, 5], the pressure and density variations in a barotropic medium are related linearly:

$$p_\Delta = \rho_\Delta c^2 . \tag{2}$$

The differential pressure p_Δ, the differential density ρ_Δ, the velocity $\mathbf{v}$, and the density of the extraneous force $\mathbf{f}_s$ depend on the coordinates of the point in the PLR and, generally speaking, differ from the values $p_{\Delta 0} \cong 0$, $\rho_{\Delta 0} \cong 0$, $\mathbf{v}_0 \cong 0$, and $\mathbf{f}_{s0} \cong 0$ far from the PLR.

The indication of process locality is assumed to be the possibility of determining the process localization such that for arbitrary small positive quantities p', ρ', v', and f_s' the inequalities

$$|\rho_\Delta| < \rho' , \ |p_\Delta| < p' , \ v < v' , \ f_s < f_s' \tag{3}$$

are fulfilled at each point of the space beyond the PLR. Obviously, for fixed ρ', p', v', and f_s' these inequalities are also fulfilled for an arbitrary expansion of the PLR. As a consequence, the shape of the expanded PLR can be arbitrary and can be chosen from considerations of convenience. We usually denote such an expanded region by D.

If a local process and its PLR shift in the medium, we additionally introduce a mobile coordinate system and tie it to the PLR (Fig. 1). The direction of the process shift in the absolute system is taken to be the x axis; the x axes in the two systems coincide. Since the velocity of the process shift in the absolute coordinate system is constant $\mathbf{v}(D)$ = const, the mobile system moves in the absolute system with exactly the same velocity. Thus,

$$v_y(D) = v_z(D) = 0 , \ \ v_x(D) = v(D) .$$

Let a certain point have mobile coordinates (x_D, y_D, z_D) and absolute coordinates (x, y, z) at the instant of time t. We write $\mathbf{v}_D\,(x_D, y_D, z_D)$ for the mobile velocity at this point and $\mathbf{v}\,(x, y, z, t)$ for the absolute velocity. Then,

$$\mathbf{v}_D\,(x_D, y_D, z_D) = \mathbf{v}\,(x, y, z, t) + \mathbf{v}_m , \tag{4}$$

where $\mathbf{v}_m = -\mathbf{v}(D)$ is the mobile velocity of the material far from the PLR. In the mobile system, the physical characteristics of a steady process are time-invariable, as are $\mathbf{v}_D$ (x_D, y_D, z_D) and $\mathbf{v}_m$.

The following general characteristics of a soliton were obtained by integrating the corresponding local characteristics over the volume V or the surface S of the PLR [6]:

$m_\Delta \equiv \int_V \rho_\Delta dV$, differential mass;

$P_\Delta \equiv \int_V p_\Delta dV$, integral differential pressure;

$\mathbf{K} \equiv \int_V \rho_\Delta \mathbf{v} dV$, momentum;

$E_k \equiv \frac{1}{2} \int_V \rho v^2 dV = E_{kx} + E_{ky} + E_{kz}$, total kinetic energy;

$E_{kx} \equiv \frac{1}{2} \int_V \rho v_x^2 dV$, $E_{ky} \equiv \frac{1}{2} \int_V \rho v_y^2 dV$, and $E_{kz} \equiv \frac{1}{2} \int_V \rho v_z^2 dV$, kinetic energy in the x, y, and z directions;

$E_{pt} \equiv \frac{1}{2c^2\rho_0} \int_V p_\Delta^2 dV$, potential energy;

$E_b \equiv m_\Delta v^2(D)/2$, kinetic energy of the equivalent solid body;

$F_p \equiv \int_S p d\mathbf{S} = \int_S p d\mathbf{S} - p_0 \int_S d\mathbf{S} = \int_S p_\Delta d\mathbf{S}$, resultant (principal vector) of the pressure force ($d\mathbf{S}$ is an element of the surface S directed into the volume); for a closed surface, $\int_S d\mathbf{S} = 0$ [5];

$\mathbf{F}_s \equiv \int_V \mathbf{f}_s dV$, resultant (principal vector) of the extraneous force;

$\mathbf{F}_i \equiv \int_V \rho \frac{d\mathbf{v}}{dt} dV$, resultant (principal vector) of the inertial force of the fluid.

The characteristics m_Δ, $\mathbf{K}$, E_k, E_{pt}, $\mathbf{F}_p$, $\mathbf{F}_s$, and $\mathbf{F}_i$ for a certain flow volume and the corresponding equations for determining them are well-known [4, 5], whereas P_Δ and E_b were first introduced by us [6]. They have a specific physical meaning and do not need explanation. The quantities m_Δ, P_Δ, $\mathbf{K}$, E_k, and E_{pt} determine the process rate, and the resultants of the forces $\mathbf{F}_p$, $\mathbf{F}_s$, and $\mathbf{F}_i$ determine the interaction of the PLR with the medium as a whole. The potential energy E_{pt} has the second order of smallness relative to p_Δ. In many cases, $E_{pt} << E_k$ and the kinetic energy is practically the entire energy of the process.

It follows from the above definitions of the integral parameters P_Δ and m_Δ that the differential mass and the integral differential pressure of a local process in a barotropic fluid (2) are related as

$$P_\Delta = m_\Delta c^2 . \tag{5}$$

According to the soliton definition (3), there is a PLR such that the local characteristics of the medium at its boundary are practically the same as at a large distance. However, a decrease in the process rate as the distance increases does not imply that there exists a large PLR for which the integral characteristics of interaction $\mathbf{F}_p$, $\mathbf{F}_s$, and $\mathbf{F}_i$ are practically equal to zero. For example, the force of the PLR interaction with the medium $\mathbf{F}_p$ depends on the integral of the differential pressure p_Δ over the entire PLR surface S. The enlargement of the surface S in a PLR expansion can balance the decrease in p_Δ with distance.

Let us consider a sequence of values of the integral characteristics of a soliton with an infinite PLR expansion. The soliton is regarded as weakly localized if with this expansion the sequence of values of some integral characteristic of the process rate (m_Δ, P_Δ, $\mathbf{K}$, E_k, E_{pt}) does not have a finite limit or the sequence of values of some integral characteristic of the PLR interaction with the medium ($\mathbf{F}_p$, $\mathbf{F}_s$, $\mathbf{F}_i$) differs significantly from zero.

The integral characteristics of the interaction of many processes tend to zero with a PLR expansion. Such processes occur exclusively in the PLR without interacting with the medium as a whole. The soliton is regarded as strongly localized if with an the infinite expansion of the PLR its integral characteristics have limiting values, being zero for the interaction forces:

$$\mathbf{K} \to \mathbf{K}_0 , \quad m_\Delta \to m_{\Delta 0} , \quad P_\Delta \to P_{\Delta 0} , \quad \mathbf{F}_p \to 0 , \quad \mathbf{F}_s \to 0 , \quad \mathbf{F}_i \to 0 .$$

As follows from the definitions of the general characteristics of a soliton, the indicated conditions are fulfilled if with an increase in the distance $r \to \infty$ the local characteristics of the process rate diminish fairly rapidly:

$$|\rho_\Delta| < ar^e , \quad |p_\Delta| < br^e , \quad |\mathbf{v}| < cr^e , \quad \left|\frac{d\mathbf{v}}{dt}\right| < dr^e ,$$

where a, b, c, and d are certain positive constants of the corresponding dimensions, and $e < -4$.

Subsequently, to avoid formal mathematical difficulties related to the calculation of limits, we will take more stringent conditions for strong localization and assume that the PLR can be selected in such a manner that the process occurs exclusively in its volume, so that beyond the PLR the following exact equalities are fulfilled:

$$\rho_\Delta = 0 , \quad p_\Delta = 0 , \quad \mathbf{v} = 0 , \quad \mathbf{f}_s = 0 . \tag{6}$$

Integral Scalar Moment of a Force. Let a certain force (for example, the pressure force or the extraneous force) be distributed over the PLR with the volume density $\mathbf{f} = \mathbf{i}f_x + \mathbf{j}f_y + \mathbf{k}f_z$. For the first time we introduce the following general characteristic, namely, the integral scalar moment of the force

$$M = \int_V (\mathbf{f}\,\mathbf{r})\, dV = M_x + M_y + M_z , \tag{7}$$

where $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ is the radius vector of the point of application of the force, the round brackets represent the scalar product, and

$$M_x = \int_V f_x x dV , \quad M_y = \int_V f_y y dV , \quad M_z = \int_V f_z z dV$$

are the components of the integral scalar moment along the corresponding axes.

It is easy to show that the quantities M_x, M_y, and M_z depend on the adopted orientation of the axes, but their sum M is independent of it.

If a certain force is distributed over the surface S with the density $\mathbf{f}$, the integral scalar moment of the force is written in a form similar to expression (7):

$$M = \int_S (\mathbf{f}\,\mathbf{r})\, dS .$$

For the pressure force, $\mathbf{f}_p = p(\mathbf{r})d\mathbf{S}$. Hence, the integral scalar moment of the external pressure force $p(\mathbf{r})$ that acts on the side of the fluid that is located outside (inside) the closed surface S is equal to

$$M = \int_S p\,(\mathbf{r})\,(d\mathbf{S}\,\mathbf{r}) , \tag{7'}$$

where the surface element $d\mathbf{S}$ is directed, respectively, inward (outward). Here, the components of the integral scalar moment are equal, respectively, to

$$M_x = \int_S p(\mathbf{r})\, x\, dS_x , \quad M_y = \int_S p(\mathbf{r})\, y\, dS_y , \quad M_z = \int_S p(\mathbf{r})\, z\, dS_z ,$$

For the concentrated forces F_i with the points of application $\mathbf{r}_i$, Eq. (7) is simplified to the utmost:

$$M = \sum (\mathbf{F}_i\, \mathbf{r}_i))$$

The sign of the quantity M characterizes the balance of tensile and compressive forces. When $M > 0$, the material on the whole is stretched, and when $M < 0$, compressed. Thus, if equal tensile forces F are applied to the ends of a rod of length l, a direct calculation yields $M = Fl$. For compressive forces, $M = -Fl$.

Let us consider the integral scalar moment of a conservative force for a strongly localized process.

The volume distribution of the conservative force depends on its potential $\mathbf{f} = -\,\mathrm{grad}\, U = -\mathbf{i}\dfrac{\partial U}{\partial x} - \mathbf{j}\dfrac{\partial U}{\partial y} - \mathbf{k}\dfrac{\partial U}{\partial z}$. Beyond the PLR, in conformity with definition (6), $\mathbf{f} = 0$. It follows from this condition that U in the free space is the same everywhere, $U = U_0$. By subtracting the constant component it is possible to provide fulfillment of the equality $U_0 = 0$. As applied to the volume density of the pressure force[5], the pressure p is a potential. The required condition is fulfilled for the differential pressure p_Δ.

Substituting the values of the quantities into Eq. (7), we obtain

$$M = \int_V (\mathbf{f}\,\mathbf{r})\, dV = -\int_V x\frac{\partial U}{\partial x} dV - \int_V y\frac{\partial U}{\partial y} dV - \int_V z\frac{\partial U}{\partial z} dV . \tag{8}$$

The first integral of this equation can be represented as

$$M_x = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\frac{\partial U}{\partial x}\, dx\, dy\, dz =$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x\frac{\partial U}{\partial x}\, dx\right) dy\, dz = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x\, dU\right) dy\, dz , \tag{9}$$

where it is taken into account that, beyond the PLR boundary, $\mathbf{f} = 0$ by definition.

In the inner integral we perform integration by parts:

$$M_x = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(xU \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} U dx \right) dy\, dz .$$

Since $U(-\infty) = U(\infty) = 0$ by the specified condition, the first term in the brackets is equal to zero and the integral M_x can be written as

$$M_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U\, dx\, dy\, dz = \int_V U\, dV \equiv I_U .$$

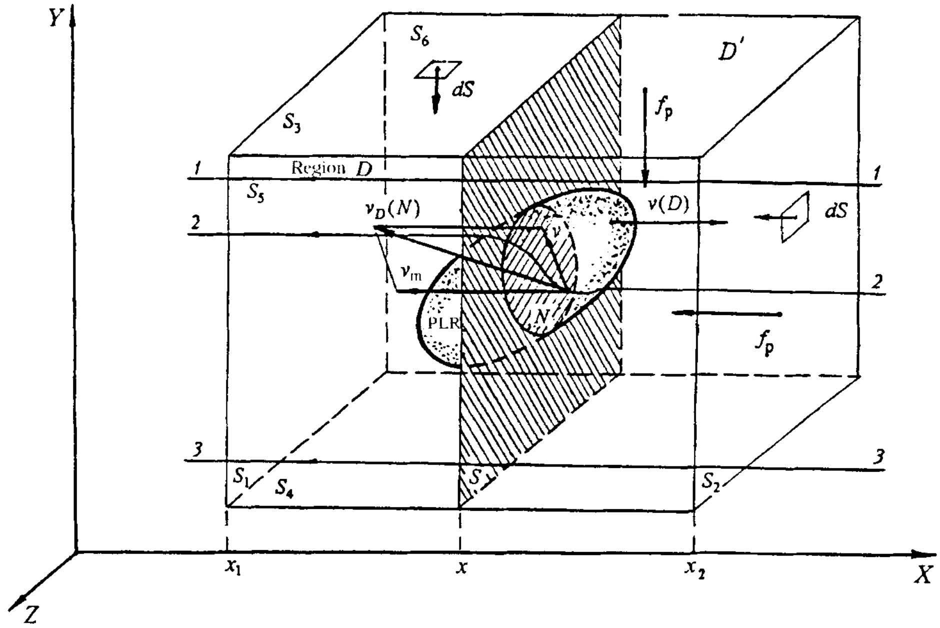

Fig. 2. Vertical cross sections of a PLR in the region D of a steady process: 1-3) streamlines in the mobile coordinate system.

The result of calculating M_y and M_z is similar:

$$M_x = M_y = M_z = I_U\,. \tag{10}$$

In particular, the components of the scalar moment of the pressure force M_p are equal to the integral differential pressure:

$$M_{px} = M_{py} = M_{pz} = P_\Delta\,. \tag{10'}$$

Thus, for the scalar moment of an extraneous conservative force, the following relation is fulfilled:

$$M_s = 3I_U\,. \tag{11}$$

which is as follows for the pressure force:

$$M_p = 3P_\Delta\,. \tag{11'}$$

Equation of Soliton Motion. We consider the motion of a local process as a whole in a continuous fluid medium. As the region D, we take an imaginary parallelepiped (Fig. 2) with edges parallel to the x, y, and z axes. The dimensions, shape, and location of the region D are such that the PLR resides entirely inside it. The region D shifts in the absolute space along with the soliton and the PLR.

In the mobile coordinate system, the PLR and the region D are motionless, and the velocity of the fluid flow is parallel to the lateral surface of the parallelepiped and orthogonal to its bases S_1 and S_2.

Equal masses flow through the bases S_1 and S_2 per unit time:

$$Q(S_1) = Q(S_2) = -\int_{S_1} \rho_0 (v_m\, dS) = -\rho_0 v_m S_1\,.$$

Let us consider the fluid flow rate for an arbitrary cross section of the region D by a plane S ($x_1 < x < x_2$) that is parallel to the bases S_1 and S_2. Since there is no inflow (outflow) through the lateral surface of the region D and the motion is steady, an invariable amount of material equal to $Q(S_1)$ flows through S per unit time:

$$Q(S_1) = Q(S) = -\int_S \rho\,(v_D dS) = -\int_S (\rho_0 + \rho_\Delta)(v_m + v_x)\, dS\,,$$

where v_x is the x-component of the absolute velocity; expression (4) was used in substitutions in the second equation.

Elementary manipulations yield

$$Q(S) = -\rho_0 v_m S - v_m \int_S \rho_\Delta dS - \int_S \rho v_x dS\,.$$

Taking into account that $S = S_1$ and $Q(S) = Q(S_1)$ and comparing the preceding two equations, we obtain

$$\int_S \rho v_x dS = -v_m \int_S \rho_\Delta dS\,. \tag{12}$$

This equation is fulfilled for arbitrary x. We next integrate Eq. (12) over x from x_1 to x_2:

$$\int_{x_1}^{x_2} \int_S \rho v_x dS\, dx = -v_m \int_{x_1}^{x_2} \int_S \rho_\Delta\, dS\, dx\,.$$

Successive integrals over S and x on the two sides of the equation are equivalent to integrals over the volume of the region D:

$$\int_D \rho v_x dV = -v_m \int_D \rho_\Delta\, dV\,.$$

The integral on the left-hand side of this equation is the x-component of the soliton momentum and the integral on the right is the differential mass of the soliton:

$$K_x = m_\Delta v_x(D)\,. \tag{13}$$

Apart from the cross sections that are orthogonal to the x axis, it is also possible to consider the cross sections of the region D that are orthogonal to the y and z axes and to obtain corresponding equations for the momentum components:

$$K_y = m_\Delta v_y(D) = 0\,,\quad K_z = m_\Delta v_z(D) = 0\,. \tag{14}$$

Since the coordinate system is selected so that the y- and z-components of the soliton velocity are equal to zero, $v_y(D) = v_z(D) = 0$, from Eqs. (13) and (14) we have the equation

$$\mathbf{K} = m_\Delta\, \mathbf{v}(D)\,. \tag{15}$$

Equation of motion (15) relates the momentum, differential mass, and velocity of the soliton. The form of the equation is similar to the relation for a solid body. However, the differential mass can be negative. In this case, the soliton velocity and momentum are antiparallel.

It should be noted that the equation of motion was obtained under the fairly general assumptions of material continuity and strong localization and stationarity of the process for an arbitrary volume distribution of the acting forces.

Equations of Force Equilibrium of a Soliton. The process is assumed to be strongly localized. As in the previous section, we fix the region D in the form of a parallelepiped oriented along the x, y, and z axes (Fig. 2). In the right side of the region D we isolate a certain part D' that is also in the form of a parallelepiped. The latter is bounded by the bases S and S_2 ($S \| S_1 \| S_2$) and the corresponding parts of the lateral faces S_3, S_4, S_5, and S_6 of the main parallelepiped. The base S is located between S_1 and S_2 and has an arbitrary coordinate x ($x_1 < x < x_2$). Like the entire region D, its selected part D' is motionless in the mobile coordinate system. The base S_2 and the lateral faces S_3, S_4, S_5, and S_6 are outside the PLR and the parameters of the material at them are the same as in the continuous medium as a whole. Corresponding to the coordinate x, the base S can dissect the PLR, and here the parameters of the material at this base can differ from the parameters of the medium in the free space far from the PLR.

We next consider the force equilibrium of D'. Since the process is steady, from the Euler theorem it follows that the x-component of the principal vector of the forces that act on the fluid inside D' is equal to zero:

$$F_{\text{par}x} = F_{\text{p}x} + F_{\text{s}x} + F_{\text{i}x} = 0\,, \tag{16}$$

where the individual terms correspond to the pressure force, extraneous force, and inertial force of the flowing masses.

The principal vector of the pressure force is determined by the integral over the surface S_{par} of the parallelepiped D':

$$\mathbf{F}_{\text{p}} = \int_{S_{\text{par}}} p\, d\mathbf{S} = \int_{S_{\text{par}}} p\, d\mathbf{S} - p_0 \int_{S_{\text{par}}} d\mathbf{S} = \int_{S_{\text{par}}} p_\Delta\, d\mathbf{S} = \int_S p_\Delta\, d\mathbf{S}\,,$$

where the surface element $d\mathbf{S}$ is oriented into the volume. Here, we used the well-known relation $\int_{S_{\text{par}}} d\mathbf{S} = 0$ for an arbitrary closed surface[5] and the equality $p_\Delta = 0$ on S_2, S_3, S_4, S_5, and S_6. Therefore,

$$F_{\text{p}x} = \int_S p_\Delta\, dS\,. \tag{17}$$

The principal vector of the extraneous force is defined by the integral over the volume V of the parallelepiped D':

$$\mathbf{F}_{\text{s}} = \int_V \mathbf{f}_{\text{s}}\, dV\,. \tag{18}$$

The principal vector of the inertial force is written as [5, 6]

$$\mathbf{F}_{\text{i}} = \int_{S_{\text{par}}} \rho \mathbf{v}_D\, (\mathbf{v}_D d\mathbf{S}) = \int_S \rho \mathbf{v}_D\, (\mathbf{v}_D d\mathbf{S}) + \int_{S_2} \rho \mathbf{v}_D\, (\mathbf{v}_D d\mathbf{S})\,.$$

Integration over the faces S_3 ... S_6 is excluded here, since its result is obviously equal to zero because the directions of the surface element and the fluid velocity at these faces are orthogonal, $d\mathbf{S} \perp \mathbf{v}_D$, so that $(\mathbf{v}_D d\mathbf{S}) = 0$.

On the face S the element $d\mathbf{S}$ is parallel to the x axis and on S_2 antiparallel. Therefore, the equation $(\mathbf{v}_D d\mathbf{S}) = v_{Dx} dS$ is fulfilled on S, and $(\mathbf{v}_D d\mathbf{S}) = -v_{Dx} dS$ on S_2, with $v_{Dx} = -v_{\text{m}}$.

Hence, the x-component of the inertial force is

$$F_{\text{i}x} = \int_S (\rho v_{Dx}^2 - \rho_0 v_{\text{m}}^2)\, dS\,.$$

We now carry out the substitutions $\rho = \rho_0 + \rho_\Delta$, $v_{Dx} = v_x + v_m$

$$F_{ix} = \int_S ((\rho_0 + \rho_\Delta)(v_x + v_m)^2 - \rho_0 v_m^2)\, dS = 2v_m \int_S \rho v_m dS + \int_S \rho v_x^2 + v_m^2 \int_S \rho_\Delta dS\,.$$

Substitution of expression (12) into the first term results in

$$F_{ix} = \int_S \rho v_x^2 dS - v_m^2 \int_S \rho_\Delta dS\,. \tag{19}$$

Using Eqs. (16)-(19), we arrive at the following equation:

$$\int_S p_\Delta dS + \int_V f_{sx} dV + \int_S \rho v_x^2 dS - v_m^2 \int_S \rho_\Delta dS = 0\,.$$

Since this equation is fulfilled for an arbitrary coordinate x of the cross section S, it can be integrated with respect to x:

$$\int_{x_1}^{x_2} \int_S p_\Delta dS dx + \int_{x_1}^{x_2} \int_V f_{sx} dV dx + \int_{x_1}^{x_2} \int_S \rho v_x^2 dS dx - v_m^2 \int_{x_1}^{x_2} \int_S \rho_\Delta dS dx = 0\,. \tag{20}$$

The first, third, and fourth integrals are, in fact, integrals over the volume of the region D that are equal to P_Δ, $2E_{kx}$, and $-m_\Delta v_m^2 = -2E_b$, respectively.

The integration over the volume of the parallelepiped D' in the second term of expression (20) is represented in the form of successive integrations: over an arbitrary cross section S that is orthogonal to the x axis and then along the coordinate x' that corresponds to the position of this cross section and lies between x and x_2:

$$J_2 = \int_{x_1}^{x_2} \int_V f_{sx} dV dx = \int_{x_1}^{x_2} dx \left(\int_x^{x_2} dx' \int_S f_{sx} dS \right). \tag{21}$$

We integrate by parts the outer integral with respect to the variable x:

$$J_2 = \int_{x_1}^{x_2} dx \left(\int_x^{x_2} dx' \int_S f_{sx} dS \right) = x \left(\int_x^{x_2} dx' \int_S f_{sx} dS \right) \Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} x d \left(\int_x^{x_2} dx' \int_S f_{sx} dS \right).$$

We next expand the first term of the difference

$$x \left(\int_x^{x_2} dx \int_S f_{sx} dS \right) \Bigg|_{x_1}^{x_2} = x_2 \left(\int_{x_2}^{x_2} dS \int_S f_{sx} dS \right) - x_1 \left(\int_{x_1}^{x_2} dx \int_S f_{sx} dS \right).$$

Since in the outer integral of the first term the limits are $x_2 = x_2$, it is equal to zero. After simple manipulations in the second term we write

$$\int_{x_1}^{x_2} dx \int_S f_{sx} dS = \int_{x_1}^{x_2} \int_S f_{sx} dS dx = \int_D f_{sx} dV = F_{sx} = 0\,.$$

Here, from definition (6) for a strongly localized process it follows that the x-component of the principal vector of the extraneous force is equal to zero.

Therefore,

$$J_2 = -\int_{x_1}^{x_2} x d\left(\int_{x}^{x_2} dx' \int_S f_{sx} dS\right) = \int_{x_1}^{x_2} x\, dx \int_S f_{sx} dS\,.$$

Differentiation of the inner integral with respect to the variable lower limit led to a change of sign. After simple manipulations,

$$J_2 = \int_{x_1}^{x_2} \int_S x f_{sx} dS dx = \int_D x f_{sx} dV = M_{sx}. \tag{21'}$$

Thus, J_2 is the x-component of the scalar moment of the extraneous force. Collecting the values of all terms in Eq. (20), we obtain the equation of equilibrium of a local process along the x axis

$$P_\Delta + M_{sx} + 2E_{kx} - 2E_b = 0\,. \tag{22}$$

Considering the cross sections of the region D that are orthogonal to the y and z axes we arrive at the corresponding equilibrium equations

$$P_\Delta + M_{sy} + 2E_{ky} = 0\,, \tag{23}$$

$$P_\Delta + M_{sz} + 2E_{kz} = 0\,. \tag{23'}$$

In these equations, the fourth term (E_b) is absent because, in accordance with the selected system of coordinates, $v_y(D) = v_z(D) = 0$.

The global equation of force equilibrium for a spatial (three-dimensional) local process is derived by adding the equations of equilibrium along the x, y, and z axes:

$$3P_\Delta + M_s + 2E_k - 2E_b = 0\,. \tag{24}$$

The global equation of force equlibrium for a two-dimensional (plane) local process is obtained by adding equilibrium equations (22) and (23) for two coordinate axes:

$$2P_\Delta + M_s + 2E_k - 2E_b = 0\,. \tag{24'}$$

The global equilibrium equation is the most general equation that involves the integral force, dynamic, and energy parameters of a soliton. In combination with the conditions of equilibrium along the coordinates, they, of course, do not replace the equation of equilibrium at a point or the equation of equilibrium for an arbitrary volume. Nonetheless, they open up broad opportunities for assessment and classification of processes and offer insight into their essence in steady and quasisteady existence. It should be noted that they were obtained using only the assumptions of continuity and homogeneity of the medium and strong localization of the process and the acting forces. With account for Eq. (5), a number of equivalent forms of the equilibrium equations can be obtained for a barotropic fluid. For example, for the global equilibrium equation for a three-dimensional process

$$3m_\Delta c^2 + M_s + 2E_k - 2E_b = 0\,.$$

Example 1. A motionless plane vortex soliton in a barotropic medium. We assume that the soliton is centrally symmetric, the fluid velocity depends only on the distance from the central point $v = v(r)$ (Fig. 3),

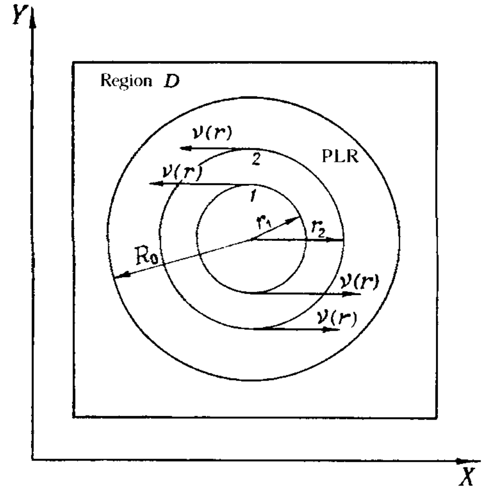

Fig. 3. Motionless vortex soliton: 1, 2) streamlines.

$v(r) = 0$ for $r > R_0$ (R_0 is the radius of the region involved in the process), and the extraneous force is absent, and therefore, $M_s = 0$. From the process immobility it follows that $E_b = 0$.

Using (5) and (24), we write the condition for global equilibrium in the form

$$E_k = -m_\Delta c^2 .$$

The analogy of this equation and the well-known Einstein formula appears to be interesting. It should be noted that this equation is fulfilled for an arbitrary velocity-radius relation $v(r)$.

Example 2. A motionless spatial soliton in the form of a ring in a barotropic fluid medium that is on the whole motionless. Such rings are frequently observed in air (smoke rings or effusion of fumes in the form of rings from flues of plants).

We assume that the ring is of centrally symmetric circular shape and its cross section is also circular (Fig. 4), with air moving only inside the ring and the extraneous force being absent, and therefore, $M_s = 0$.

The velocity at each point inside the ring has two components. One of them (v_R) is directed around a large circle, and the other (v_r), around a small circle. From the equation of force equilibrium for one of the axes it follows that along the ring

$$2E_{kR} + P_\Delta = 0 ,$$

where E_{kR} is the energy of the velocity component along the ring.

As follows from Eqs. (23) and (23′), the equation of force equilibrium in the ring cross section appears as

$$E_{kr} + P_\Delta = 0 ,$$

where E_{kr} is the total energy of two transverse velocity components. Hence we have a relation between the rms values of the longitudinal and transverse wind velocities:

$$\hat{v}_{kR} = \frac{1}{\sqrt{2}} \hat{v}_{kr} .$$

The wind velocity along the ring is $\sqrt{2}$-fold smaller than the transverse velocity. This relation is fulfilled on the average over the cross section. Here, the total kinetic energy is connected to the differential mass of the ring by a relation that differs from the analogous equation for a plane process by the coefficient:

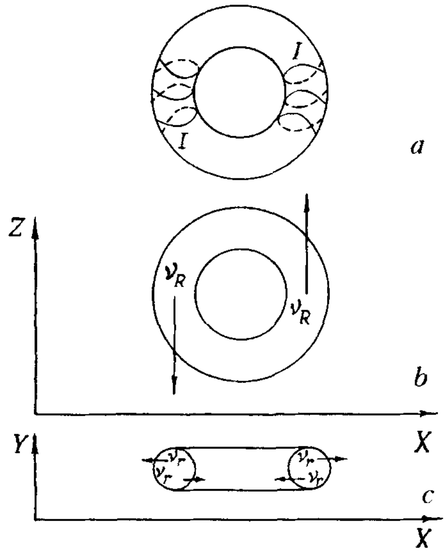

Fig. 4. Motionless ring: a) axonometry; b) longitudinal section; c) cross section; *I* denotes streamlines.

$$E_k = -\frac{3}{2} m_\Delta c^2 .$$

Conclusion. The motion and force equilibrium of a local steady process (a soliton) in a continuous fluid medium have been considered. An equation of motion has been obtained. According to this equation, the overall momentum of a soliton is equal to the product of its differential mass (the total excess mass in its volume as compared with the same volume of the free space) and the motion velocity in the medium. With a positive excess mass, the soliton momentum and velocity are parallel, and with a negative excess mass (i.e., mass deficiency), counterparallel.

Also, we have obtained a simple, in form, global equation of force equilibrium and simple, in form, equations of force equilibrium of a soliton along individual coordinate axes, which include integral characteristics of it over the volume of the region involved in the process. Among these characteristics are the total kinetic energy of the soliton, the kinetic energy of the equivalent solid body, the integral excess pressure over the soliton volume (a deficiency when the excess is negative), and a new characteristic for the extraneous (external) force, namely, the integral scalar moment. We have demonstrated the possibilities of using the proposed theory in calculating the characteristics of specific local processes based on generalized information on their structure.

The equation of motion, the global equation of force equilibrium, and the equations of equilibrium along the coordinates do not, of course, replace the continuity equation or the Euler equation for force equilibrium at a point. Nonetheless, they open up broad opportunities for assessment and classification of processes and offer insight into their essence in steady and quasisteady existence.

NOTATION

p, pressure; ρ, mass density; p_Δ and ρ_Δ, differences between the fluid pressure and density at an arbitrary point and in the free space; $\mathbf{v}(D)$, absolute velocity of the soliton; $\mathbf{v}$, absolute velocity of the fluid; $\mathbf{v}_D$, mobile velocity of the fluid; $\mathbf{v}_m$, mobile velocity of the fluid in the free space; $\mathbf{f}_s$, volume density of the extraneous force; $\mathbf{f}_i$, volume density of the inertial force; $\mathbf{f}_p$, volume density of the pressure force; c, velocity of sound; PLR, process localization region; D, arbitrary expansion of the PLR; V, PLR volume; S, PLR surface; m_Δ, differential mass of the soliton; P_Δ, integral differential pressure of the soliton; $\mathbf{K}$, soliton momentum; E_k, kinetic energy of the soliton; E_{kx}, E_{ky}, and E_{kz}, kinetic energy of the soliton along the x, y, and z axes; E_{pt}, potential energy of the soliton; E_b, kinetic energy of the equivalent solid body; **r**, radius vector; M, total integral scalar moment; M_x, M_y, and M_z, integral scalar moments along the x, y, and z axes, respectively; M_s, total integral scalar moment for the extraneous force; $\mathbf{F}_p$, $\mathbf{F}_s$, and $\mathbf{F}_i$, resultants of the pressure force, extraneous force, and inertial force, respectively. Subscripts: Δ, differential quantity; D, in the mobile coordinate system; m, mobile; s, extraneous; i, inertia; p, pressure; k, kinetic; x, y, z, coordinate axes; pt, potential; b, solid body; 0, undisturbed.

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